Log smooth, étale and differentials

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– Log smooth and étale –

Recall that a log scheme (X, M) is fine if it is coherent and integral. That is, étale locally on X, there exists a finitely generated integral monoid P and morphism $P_X \to \mathcal{O}_X$ (where P_X denotes the constant sheaf) whose associated log structure is isomorphic to M.

Definition 1. Let $i : (X, M) \to (Y, N)$ be a morphism of fine log schemes. We say i is a closed immersion if the underlying map of schemes $X \to Y$ is a closed immersion, and $i^*N \to M$ is surjective. If the latter is an isomorphism (i.e. i is strict), we call i an exact closed immersion.

Definition 2. Let $f : (X, M) \to (Y, N)$ be a morphism of fine log schemes. We say f is formally smooth (resp. formally étale) if for any commutative diagram

$$\begin{array}{ccc} (T', L') & \longrightarrow & (X, M) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ (T, L) & \longrightarrow & (Y, N) \end{array}$$

with i an exact closed immersion, and $T' \subset T$ defined by an ideal I s.t. $I^2 = 0$, there exists étale locally on T (resp. a unique) $g : (T, L) \to (X, M)$ such that everything commutes.

We say that f is smooth (resp. étale) if it is formally smooth (resp. formally étale) and the underlying morphism of schemes is locally of finite presentation.

Proposition 3. *A strict morphism of fine log schemes $(X, M) \to (Y, N)$ is smooth (resp. étale) if and only if the underlying morphism of schemes is so.*

Proof. Assuming the morphism is smooth (resp. étale), to prove it is smooth (resp. étale) as a morphism of schemes one simply puts $L = \mathcal{O}_T$ on T, and $L' = i^*\mathcal{O}_T$ on T'. Conversely, we have existence (resp. and uniqueness) as morphisms of schemes, and there is a unique choice to make it a morphism of log schemes. □

The standard example of a smooth or étale morphism is given by the following proposition.

Proposition 4. *Let P and Q be f.g. integral monoids, $Q \to P$ a homomorphism, R a ring, such that $\ker(Q^{gp} \to P^{gp})$ and (resp. the torsion part of) coker$(Q^{gp} \to P^{gp})$ are finite groups whose order is invertible in R. Let*

$$X = \operatorname{Spec}(R[P]) \qquad \textit{and} \qquad Y = \operatorname{Spec}(R[Q])$$

and consider their canonical log structures M and N. Then the morphism $(X, M) \to (Y, N)$ is étale (resp. smooth).

Proof. Consider a diagram as in Definition 2. Since $I^2 = 0$, we have an embedding

$$I \to \mathcal{O}_T^* \subset L \qquad x \mapsto 1 + x.$$

We have a cartesian diagram:

$$\begin{array}{ccc} L & \longrightarrow & L/I = L' \\ \downarrow & & \downarrow \\ L^{\mathrm{gp}} & \longrightarrow & L^{\mathrm{gp}}/I = (L')^{\mathrm{gp}} \end{array}$$

By the assumptions on the kernel and cokernel of $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$, there exists étale locally a dashed arrow:

$$\begin{array}{ccc} (L')^{\mathrm{gp}} & \longleftarrow & P^{\mathrm{gp}} \\ \uparrow & \swarrow & \uparrow \\ L^{\mathrm{gp}} & \longleftarrow & Q^{\mathrm{gp}} \end{array}$$

By the cartesian diagram above, this induces a map $P \to L$, which induces the desired $(T, L) \to (X, M)$. $\square$

The following theorem shows us why this is the standard example of a smooth or étale morphism.

Theorem 5. *Let $f : (X, M) \to (Y, N)$ be a morphism of fine log schemes. Assume we are given a chart $Q_Y \to N$ of N. Then the following are equivalent:*

(i) f is étale (resp. smooth)

(ii) étale locally on X, there exists a chart $(P_X \to M, Q_Y \to N, Q \to P)$ of f extending $Q_Y \to N$, satisfying

(a) the kernel and (resp. torsion part of the) cokernel of $Q^{gp} \to P^{gp}$ are finite groups of order invertible on X,

(b) the induced morphism of schemes

$$X \to Y \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P]$$

is étale. (For the smooth part of the theorem, one can equivalently require this map to be smooth instead.)

Example 6 (Log smooth curve)**.** Let $X = \operatorname{Spec} k[x, y]/(xy)$ and $Y = \operatorname{Spec} k$, and take M and N to be the log structure on X and Y induced by

$$P = \mathbb{N}^2 \to k[x, y]/(xy) : (a, b) \mapsto x^a y^b \qquad \text{and} \qquad Q = \mathbb{N} \to k : a \mapsto 0^a,$$

respectively. Indeed these provide (global) charts for the log structures. Let $f : (X, M) \to (Y, N)$ be the morphism induced by $\Delta : \mathbb{N} \to \mathbb{N}^2 : a \mapsto (a, a)$. Indeed Δ induces a map $h : f^{-1}N = \underline{\mathbb{N} \oplus k^*} \to M$ given by $(a, u) \mapsto 0^a u$, and the following diagram commutes:

$$\begin{array}{ccc} \underline{\mathbb{N} \oplus k^*} & \xrightarrow{h} & M \\ {\scriptstyle f^{-1}\beta}\downarrow & & \downarrow{\scriptstyle \alpha} \\ k & \longrightarrow & \mathcal{O}_X \end{array}$$

We will now show that f is smooth as log schemes using Theorem 5. (a) Indeed $\ker(\Delta^{\mathrm{gp}}) = (0)$ and $(\mathrm{coker}(\Delta^{\mathrm{gp}}))_{\mathrm{torsion}} = (0)$ are finite and their order $(= 1)$ is invertible on X. (b) The induced morphism of schemes

$$X \longrightarrow \operatorname{Spec} k \times_{\operatorname{Spec} k[t]} \operatorname{Spec} k[x,y] \simeq k[x,y]/(xy)$$

is the identity, in particular étale.

Example 7 (Toroidal embedding)**.** Let k be a field, and X a scheme locally of finite type over k, with fine log structure M. Then Theorem 5 says (X, M) is smooth over $\operatorname{Spec} k$ (with trivial log structure, take chart with $Q = \{1\}$) if and only if étale locally on X, there exists a f.g. integral monoid P and étale morphism $X \to \operatorname{Spec}(k[P])$ such that $(P_X)^a \simeq M$ and the torsion part of P^{gp} is finite of order invertible in k.

Hence, such (X, M) corresponds to a toroidal embedding, which is étale locally given by the open immersion

$$X \times_{\operatorname{Spec}(k[P])} \operatorname{Spec}(k[P^{\mathrm{gp}}]) \subset X.$$

– Log differentials –

Definition 8. Let $f : (X, M) \to (Y, N)$ be a morphism of log schemes. Then we define the $\mathcal{O}_X$-module $\omega^1_{X/Y}$ of log differentials to be the quotient of

$$\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M^{\mathrm{gp}})$$

by the following relations of local sections:

(i) $(d\alpha(a), 0) = (0, \alpha(a) \otimes a)$ for all $a \in M$

(ii) $(0, 1 \otimes a) = 0$ for all $a \in \mathrm{im}(f^{-1}N \to M)$

We think of $\omega^1_{X/Y}$ as an extension of the usual sheaf of differentials $\Omega^1_{X/Y}$. Elements of the form $(0, 1 \otimes a)$ we also write as $d\log(a)$. This notation makes sense in light of relation (i) as $d\log(a) = \frac{da}{a}$. Basically, we add symbols "$d\log(a)$" for all $a \in M^{\mathrm{gp}}$ (up to the image of $f^{-1}N$). Note that we have a homomorphism

$$d\log : M \to \omega^1_{X/Y} : f \mapsto d\log(f).$$

Remark 9. The map

$$\mathcal{O}_X \otimes_{\mathbb{Z}} M^{\mathrm{gp}} \to \omega^1_{X/Y} : a \otimes b \mapsto a \cdot d\log(b)$$

is surjective: given any df, locally we can choose f to be invertible, and then use that $df = f \cdot d\log(f)$.

Example 10. Let $X = \operatorname{Spec}(R[P])$ and $Y = \operatorname{Spec}(R[Q])$ with canonical log structures, and a map between them induced by a homomorphism of monoids $Q \to P$. Then

$$\begin{array}{ccc} \mathcal{O}_X \otimes_{\mathbb{Z}} (P^{\mathrm{gp}}/\,\mathrm{im}(Q^{\mathrm{gp}})) & \simeq & \omega^1_{X/Y} \\ a \otimes b & \mapsto & a \cdot d\log(b). \end{array}$$

Example 11. Let k be a field with characteristic $\neq 2$. Consider the map $\mathbb{A}^1_k \to \mathbb{A}^1_k$ given by $t = s^2$. The usual sheaf of differentials $\Omega^1_{\mathbb{A}^1_k}$ is given by the free $k[x]$-module with generator dx, and the above map induces

$$\Omega^1_{\mathbb{A}^1_k} \to \Omega^1_{\mathbb{A}^1_k} \qquad\qquad dt \mapsto 2sds$$

However, this map is not very nice in the sense that the cokernel has support at $s = 0$. The idea is that we 'fix' this using log differentials.

Take the divisor $D = \{0\}$, and consider the log structure M on $\mathbb{A}^1_k$ given by

$$M = \{f \in \mathcal{O}_X : f \text{ is invertible outside } D\}.$$

Note that $x \in M$, so we obtain $d\log(x) = \frac{dx}{x} \in \omega^1_{\mathbb{A}^1_k}$. In fact, one can show that $\omega^1_{\mathbb{A}^1_k}$ is given by the free $k[x]$-module generated by $\frac{dx}{x}$. Now the map from before is given by

$$\omega^1_{\mathbb{A}^1_k} \to \omega^1_{\mathbb{A}^1_k} \qquad\qquad \frac{dt}{t} \mapsto 2\frac{ds}{s},$$

which is now an isomorphism.

Proposition 12. *Let $f : (X, M) \to (Y, N)$ be a smooth morphism of fine log schemes. Then the $\mathcal{O}_X$-module $\omega^1_{X/Y}$ is locally free of finite type.*